Review of Chapter 0&1

The following list states important properties of real numbers

Property	*Example(s)*
1. $a - b = a + (-b)$	$2 - 7 = 2 + (-7) = -5$
2. $a - (-b) = a + b$	$2 - (-7) = 2 + 7 = 9$
3. $-a = (-1)(a)$	$-7 = (-1)(7)$
4. $a(b + c) = ab + ac$	$6(7 + 2) = 6 \cdot 7 + 6 \cdot 2 = 54$
5. $a(b - c) = ab - ac$	$6(7 - 2) = 6 \cdot 7 - 6 \cdot 2 = 30$
6. $-(a + b) = -a - b$	$-(7 + 2) = -7 - 2 = -9$
7. $-(a - b) = -a + b$	$-(2 - 7) = -2 + 7 = 5$
8. $-(-a) = a$	$-(-2) = 2$
9. $a(0) = 0$	$2(0) = 0$
10. $(-a)(b) = -(ab) = a(-b)$	$(-2)(7) = -(2 \cdot 7) = 2(-7) = -14$
11. $(-a)(-b) = ab$	$(-2)(-7) = 2 \cdot 7 = 14$
12. $\frac{a}{1} = a$	$\frac{7}{1} = 7, \frac{-2}{1} = -2$
13. $\frac{a}{b} = a\left(\frac{1}{b}\right)$ for $b \neq 0$	$\frac{2}{7} = 2\left(\frac{1}{7}\right)$

Property	*Example(s)*
14. $\dfrac{a}{-b} = -\dfrac{a}{b} = \dfrac{-a}{b}$ for $b \neq 0$	$\dfrac{2}{-7} = -\dfrac{2}{7} = \dfrac{-2}{7}$
15. $\dfrac{-a}{-b} = \dfrac{a}{b}$ for $b \neq 0$	$\dfrac{-2}{-7} = \dfrac{2}{7}$
16. $\dfrac{0}{a} = 0$ for $a \neq 0$	$\dfrac{0}{7} = 0$
17. $\dfrac{a}{a} = 1$ for $a \neq 0$	$\dfrac{2}{2} = 1, \dfrac{-5}{-5} = 1$
18. $a\left(\dfrac{b}{a}\right) = b$ for $a \neq 0$	$2\left(\dfrac{7}{2}\right) = 7$
19. $a \cdot \dfrac{1}{a} = 1$ for $a \neq 0$	$2 \cdot \dfrac{1}{2} = 1$
20. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ for $b, d \neq 0$	$\dfrac{2}{3} \cdot \dfrac{4}{5} = \dfrac{2 \cdot 4}{3 \cdot 5} = \dfrac{8}{15}$
21. $\dfrac{ab}{c} = \left(\dfrac{a}{c}\right)b = a\left(\dfrac{b}{c}\right)$ for $c \neq 0$	$\dfrac{2 \cdot 7}{3} = \dfrac{2}{3} \cdot 7 = 2 \cdot \dfrac{7}{3}$
22. $\dfrac{a}{bc} = \dfrac{a}{b} \cdot \dfrac{1}{c} = \dfrac{1}{b} \cdot \dfrac{a}{c}$ for $b, c \neq 0$	$\dfrac{2}{3 \cdot 7} = \dfrac{2}{3} \cdot \dfrac{1}{7} = \dfrac{1}{3} \cdot \dfrac{2}{7}$
23. $\dfrac{a}{b} = \dfrac{a}{b} \cdot \dfrac{c}{c} = \dfrac{ac}{bc}$ for $b, c \neq 0$	$\dfrac{2}{7} = \left(\dfrac{2}{7}\right)\left(\dfrac{5}{5}\right) = \dfrac{2 \cdot 5}{7 \cdot 5}$

Property	Example(s)
24. $\dfrac{a}{b(-c)} = \dfrac{a}{(-b)(c)} = \dfrac{-a}{bc} = \dfrac{-a}{(-b)(-c)} = -\dfrac{a}{bc}$ for $b, c \neq 0$	$\dfrac{2}{3(-5)} = \dfrac{2}{(-3)(5)} = \dfrac{-2}{3(5)} = \dfrac{-2}{(-3)(-5)} = -\dfrac{2}{3(5)} = -\dfrac{2}{15}$
25. $\dfrac{a(-b)}{c} = \dfrac{(-a)b}{c} = \dfrac{ab}{-c} = \dfrac{(-a)(-b)}{-c} = -\dfrac{ab}{c}$ for $c \neq 0$	$\dfrac{2(-3)}{5} = \dfrac{(-2)(3)}{5} = \dfrac{2(3)}{-5} = \dfrac{(-2)(-3)}{-5} = -\dfrac{2(3)}{5} = -\dfrac{6}{5}$
26. $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$ for $c \neq 0$	$\dfrac{2}{9} + \dfrac{3}{9} = \dfrac{2+3}{9} = \dfrac{5}{9}$
27. $\dfrac{a}{c} - \dfrac{b}{c} = \dfrac{a-b}{c}$ for $c \neq 0$	$\dfrac{2}{9} - \dfrac{3}{9} = \dfrac{2-3}{9} = \dfrac{-1}{9}$
28. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$ for $b, d \neq 0$	$\dfrac{4}{5} + \dfrac{2}{3} = \dfrac{4 \cdot 3 + 5 \cdot 2}{5 \cdot 3} = \dfrac{22}{15}$
29. $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{ad-bc}{bd}$ for $b, d \neq 0$	$\dfrac{4}{5} - \dfrac{2}{3} = \dfrac{4 \cdot 3 - 5 \cdot 2}{5 \cdot 3} = \dfrac{2}{15}$
30. $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}$ for $b, c, d \neq 0$	$\dfrac{\frac{2}{3}}{\frac{7}{5}} = \dfrac{2}{3} \div \dfrac{7}{5} = \dfrac{2}{3} \cdot \dfrac{5}{7} = \dfrac{2 \cdot 5}{3 \cdot 7} = \dfrac{10}{21}$

Property	*Example(s)*
31. $\dfrac{a}{\frac{b}{c}} = a \div \dfrac{b}{c} = a \cdot \dfrac{c}{b} = \dfrac{ac}{b}$ for $b, c \neq 0$	$\dfrac{2}{\frac{3}{5}} = 2 \div \dfrac{3}{5} = 2 \cdot \dfrac{5}{3} = \dfrac{2 \cdot 5}{3} = \dfrac{10}{3}$
32. $\dfrac{\frac{a}{b}}{c} = \dfrac{a}{b} \div c = \dfrac{a}{b} \cdot \dfrac{1}{c} = \dfrac{a}{bc}$ for $b, c \neq 0$	$\dfrac{\frac{2}{3}}{5} = \dfrac{2}{3} \div 5 = \dfrac{2}{3} \cdot \dfrac{1}{5} = \dfrac{2}{3 \cdot 5} = \dfrac{2}{15}$

Exponents and Radicals

1. $x^n = \underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}_{n \text{ factors}}$

2. $x^{-n} = \dfrac{1}{x^n} = \dfrac{1}{\underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}_{n \text{ factors}}}$ for $x \neq 0$

3. $\dfrac{1}{x^{-n}} = x^n$ for $x \neq 0$

4. $x^0 = 1$

EXAMPLE 1 Exponents

a. $\left(\dfrac{1}{2}\right)^4 = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{1}{16}$

b. $3^{-5} = \dfrac{1}{3^5} = \dfrac{1}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \dfrac{1}{243}$

c. $\dfrac{1}{3^{-5}} = 3^5 = 243$

d. $2^0 = 1, \pi^0 = 1, (-5)^0 = 1$

e. $x^1 = x$

The **principal nth root**[1] of x is the nth root of x that is positive if x is positive and is negative if x is negative and n is odd. We denote the principal nth root of x by $\sqrt[n]{x}$. Thus,

$$\sqrt[n]{x} \text{ is } \begin{cases} \text{positive if } x \text{ is positive} \\ \text{negative if } x \text{ is negative and } n \text{ is odd} \end{cases}$$

For example, $\sqrt[2]{9} = 3$, $\sqrt[3]{-8} = -2$, and $\sqrt[3]{\dfrac{1}{27}} = \frac{1}{3}$. We define $\sqrt[n]{0} = 0$.

The symbol $\sqrt[n]{x}$ is called a **radical.** Here n is the *index*, x is the *radicand*, and $\sqrt{}$ is the *radical sign*. With principal square roots, we usually omit the index and write $\sqrt{x}$ instead of $\sqrt[2]{x}$. Thus, $\sqrt{9} = 3$.

Here are the basic laws of exponents and radicals:[2]

Law	*Example(s)*
1. $x^m \cdot x^n = x^{m+n}$	$2^3 \cdot 2^5 = 2^8 = 256;\ x^2 \cdot x^3 = x^5$
2. $x^0 = 1$	$2^0 = 1$
3. $x^{-n} = \dfrac{1}{x^n}$	$2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$
4. $\dfrac{1}{x^{-n}} = x^n$	$\dfrac{1}{2^{-3}} = 2^3 = 8;\ \dfrac{1}{x^{-5}} = x^5$
5. $\dfrac{x^m}{x^n} = x^{m-n} = \dfrac{1}{x^{n-m}}$	$\dfrac{2^{12}}{2^8} = 2^4 = 16;\ \dfrac{x^8}{x^{12}} = \dfrac{1}{x^4}$
6. $\dfrac{x^m}{x^m} = 1$	$\dfrac{2^4}{2^4} = 1$
7. $(x^m)^n = x^{mn}$	$(2^3)^5 = 2^{15};\ (x^2)^3 = x^6$
8. $(xy)^n = x^n y^n$	$(2 \cdot 4)^3 = 2^3 \cdot 4^3 = 8 \cdot 64 = 512$
9. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$	$\left(\dfrac{2}{3}\right)^3 = \dfrac{2^3}{3^3} = \dfrac{8}{27}$
10. $\left(\dfrac{x}{y}\right)^{-n} = \left(\dfrac{y}{x}\right)^n$	$\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2 = \dfrac{16}{9}$
11. $x^{1/n} = \sqrt[n]{x}$	$3^{1/5} = \sqrt[5]{3}$

12. $x^{-1/n} = \dfrac{1}{x^{1/n}} = \dfrac{1}{\sqrt[n]{x}}$ $\qquad$ $4^{-1/2} = \dfrac{1}{4^{1/2}} = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$

13. $\sqrt[n]{x}\sqrt[n]{y} = \sqrt[n]{xy}$ $\qquad$ $\sqrt[3]{9}\sqrt[3]{2} = \sqrt[3]{18}$

14. $\dfrac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\dfrac{x}{y}}$ $\qquad$ $\dfrac{\sqrt[3]{90}}{\sqrt[3]{10}} = \sqrt[3]{\dfrac{90}{10}} = \sqrt[3]{9}$

15. $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$ $\qquad$ $\sqrt[3]{\sqrt[4]{2}} = \sqrt[12]{2}$

16. $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$ $\qquad$ $8^{2/3} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 2^2 = 4$

17. $(\sqrt[m]{x})^m = x$ $\qquad$ $(\sqrt[8]{7})^8 = 7$

EXAMPLE

Eliminate negative exponents in $\dfrac{x^{-2}y^3}{z^{-2}}$ for $x \neq 0, z \neq 0$.

Solution: $\dfrac{x^{-2}y^3}{z^{-2}} = x^{-2} \cdot y^3 \cdot \dfrac{1}{z^{-2}} = \dfrac{1}{x^2} \cdot y^3 \cdot z^2 = \dfrac{y^3z^2}{x^2}$

EXAMPLE

Simplify $\sqrt[4]{48}$.

Solution: $$\sqrt[4]{48} = \sqrt[4]{16 \cdot 3} = \sqrt[4]{16}\,\sqrt[4]{3} = 2\sqrt[4]{3}$$

Simplify $\dfrac{\sqrt{20}}{\sqrt{5}}$.

Solution: $$\frac{\sqrt{20}}{\sqrt{5}} = \sqrt{\frac{20}{5}} = \sqrt{4} = 2$$

Simplify $\sqrt[3]{x^6y^4}$.

Solution: $$\sqrt[3]{x^6y^4} = \sqrt[3]{(x^2)^3y^3y} = \sqrt[3]{(x^2)^3} \cdot \sqrt[3]{y^3} \cdot \sqrt[3]{y}$$
$$= x^2y\sqrt[3]{y}$$

EXAMPLE **Rationalizing Denominators**

$$\frac{2}{\sqrt{5}} = \frac{2}{5^{1/2}} = \frac{2 \cdot 5^{1/2}}{5^{1/2} \cdot 5^{1/2}} = \frac{2 \cdot 5^{1/2}}{5^1} = \frac{2\sqrt{5}}{5}$$

Operations with Algebraic Expressions

EXAMPLE 1 Algebraic Expressions

a. $\sqrt[3]{\dfrac{3x^3 - 5x - 2}{10 - x}}$ is an algebraic expression in the variable x.

b. $10 - 3\sqrt{y} + \dfrac{5}{7 + y^2}$ is an algebraic expression in the variable y.

c. $\dfrac{(x + y)^3 - xy}{y} + 2$ is an algebraic expression in the variables x and y.

Special Products

1. $x(y + z) = xy + xz$	distributive property
2. $(x + a)(x + b) = x^2 + (a + b)x + ab$	
3. $(ax + c)(bx + d) = abx^2 + (ad + cb)x + cd$	
4. $(x + a)^2 = x^2 + 2ax + a^2$	square of a sum
5. $(x - a)^2 = x^2 - 2ax + a^2$	square of a difference
6. $(x + a)(x - a) = x^2 - a^2$	product of sum and difference
7. $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$	cube of a sum
8. $(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$	cube of a difference

EXAMPLE 5 **Special Products**

a. By Rule 2,

$$
\begin{aligned}
(x+2)(x-5) &= [x+2][x+(-5)] \\
&= x^2 + (2-5)x + 2(-5) \\
&= x^2 - 3x - 10
\end{aligned}
$$

b. By Rule 3,

$$
\begin{aligned}
(3z+5)(7z+4) &= 3 \cdot 7z^2 + (3 \cdot 4 + 5 \cdot 7)z + 5 \cdot 4 \\
&= 21z^2 + 47z + 20
\end{aligned}
$$

c. By Rule 5,

$$
\begin{aligned}
(x-4)^2 &= x^2 - 2(4)x + 4^2 \\
&= x^2 - 8x + 16
\end{aligned}
$$

d. By Rule 6,

$$
\begin{aligned}
(\sqrt{y^2+1}+3)(\sqrt{y^2+1}-3) &= (\sqrt{y^2+1})^2 - 3^2 \\
&= (y^2+1) - 9 \\
&= y^2 - 8
\end{aligned}
$$

e. By Rule 7,

$$
\begin{aligned}
(3x+2)^3 &= (3x)^3 + 3(2)(3x)^2 + 3(2)^2(3x) + (2)^3 \\
&= 27x^3 + 54x^2 + 36x + 8
\end{aligned}
$$

Factoring

Rules for Factoring

1. $xy + xz = x(y + z)$	common factor
2. $x^2 + (a + b)x + ab = (x + a)(x + b)$	
3. $abx^2 + (ad + cb)x + cd = (ax + c)(bx + d)$	
4. $x^2 + 2ax + a^2 = (x + a)^2$	perfect-square trinomial
5. $x^2 - 2ax + a^2 = (x - a)^2$	perfect-square trinomial
6. $x^2 - a^2 = (x + a)(x - a)$	difference of two squares
7. $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$	sum of two cubes
8. $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$	difference of two cubes

EXAMPLE

a. Factor $3x^2 + 6x + 3$ completely.

Solution: First we remove a common factor. Then we factor the resulting expression completely. Thus, we have

$$\begin{aligned} 3x^2 + 6x + 3 &= 3(x^2 + 2x + 1) \\ &= 3(x + 1)^2 \qquad \text{Rule 4} \end{aligned}$$

b. Factor $x^2 - x - 6$ completely.

Solution: If this trinomial factors into the form $(x + a)(x + b)$, which is a product of two binomials, then we must determine the values of a and b. Since $(x+a)(x+b) = x^2 + (a+b)x + ab$, it follows that

$$x^2 + (-1)x + (-6) = x^2 + (a+b)x + ab$$

By equating corresponding coefficients, we want

$$a + b = -1 \quad \text{and} \quad ab = -6$$

If $a = -3$ and $b = 2$, then both conditions are met and hence

$$x^2 - x - 6 = (x-3)(x+2)$$

As a check, it is wise to multiply the right side to see if it agrees with the left side.

c. Factor $x^2 - 7x + 12$ completely.

Solution: $$x^2 - 7x + 12 = (x-3)(x-4)$$

EXAMPLE 3 Factoring

The following is an assortment of expressions that are completely factored. The numbers in parentheses refer to the rules used.

a. $x^2 + 8x + 16 = (x + 4)^2$ (4)

b. $9x^2 + 9x + 2 = (3x + 1)(3x + 2)$ (3)

c. $6y^3 + 3y^2 - 18y = 3y(2y^2 + y - 6)$ (1)

$\qquad = 3y(2y - 3)(y + 2)$ (3)

d. $x^2 - 6x + 9 = (x - 3)^2$ (5)

e. $z^{1/4} + z^{5/4} = z^{1/4}(1 + z)$ (1)

f. $x^4 - 1 = (x^2 + 1)(x^2 - 1)$ (6)

$\qquad = (x^2 + 1)(x + 1)(x - 1)$ (6)

g. $x^{2/3} - 5x^{1/3} + 4 = (x^{1/3} - 1)(x^{1/3} - 4)$ (2)

h. $ax^2 - ay^2 + bx^2 - by^2 = a(x^2 - y^2) + b(x^2 - y^2)$ (1), (1)

$\qquad = (x^2 - y^2)(a + b)$ (1)

$\qquad = (x + y)(x - y)(a + b)$ (6)

i. $8 - x^3 = (2)^3 - (x)^3 = (2 - x)(4 + 2x + x^2)$ (8)

j. $x^6 - y^6 = (x^3)^2 - (y^3)^2 = (x^3 + y^3)(x^3 - y^3)$ (6)

$\qquad = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$ (7), (8)

Fractions

EXAMPLE 1 Simplifying Fractions

a. Simplify $\dfrac{x^2 - x - 6}{x^2 - 7x + 12}$.

Solution: First, we completely factor both the numerator and the denominator:

$$\frac{x^2 - x - 6}{x^2 - 7x + 12} = \frac{(x-3)(x+2)}{(x-3)(x-4)}$$

Dividing both numerator and denominator by the common factor $x - 3$, we have

$$\frac{(x-3)(x+2)}{(x-3)(x-4)} = \frac{1(x+2)}{1(x-4)} = \frac{x+2}{x-4} \quad \text{for } x \neq 3$$

b. Simplify $\dfrac{2x^2 + 6x - 8}{8 - 4x - 4x^2}$.

Solution:

$$\frac{2x^2 + 6x - 8}{8 - 4x - 4x^2} = \frac{2(x^2 + 3x - 4)}{4(2 - x - x^2)} = \frac{2(x-1)(x+4)}{4(1-x)(2+x)}$$

$$= \frac{2(x-1)(x+4)}{2(2)[(-1)(x-1)](2+x)}$$

$$= \frac{x+4}{-2(2+x)} \quad \text{for } x \neq 1$$

EXAMPLE 2 Multiplying Fractions

$$\frac{x}{x+2} \cdot \frac{x+3}{x-5} = \frac{x(x+3)}{(x+2)(x-5)}$$

EXAMPLE 3 Dividing Fractions

a. $$\frac{x}{x+2} \div \frac{x+3}{x-5} = \frac{x}{x+2} \cdot \frac{x-5}{x+3} = \frac{x(x-5)}{(x+2)(x+3)}$$

b. $$\frac{\frac{x-5}{x-3}}{2x} = \frac{\frac{x-5}{x-3}}{\frac{2x}{1}} = \frac{x-5}{x-3} \cdot \frac{1}{2x} = \frac{x-5}{2x(x-3)}$$

EXAMPLE **Adding and Subtracting Fractions**

a. $$\frac{x^2-5x+4}{x^2+2x-3}-\frac{x^2+2x}{x^2+5x+6}=\frac{(x-1)(x-4)}{(x-1)(x+3)}-\frac{x(x+2)}{(x+2)(x+3)}$$

$$=\frac{x-4}{x+3}-\frac{x}{x+3}=\frac{(x-4)-x}{x+3}=-\frac{4}{x+3}\quad \text{for } x\neq -2, 1$$

b. $$\frac{t}{(3t+2)}-\frac{4}{t-1}=\frac{t(t-1)}{(3t+2)(t-1)}-\frac{4(3t+2)}{(3t+2)(t-1)}$$

$$=\frac{t(t-1)-4(3t+2)}{(3t+2)(t-1)}$$

$$=\frac{t^2-t-12t-8}{(3t+2)(t-1)}=\frac{t^2-13t-8}{(3t+2)(t-1)}$$

c. $$\frac{4}{q-1}+3=\frac{4}{q-1}+\frac{3(q-1)}{q-1}$$

$$=\frac{4+3(q-1)}{q-1}=\frac{3q+1}{q-1}$$

Linear Equations

Definition

A ***linear equation*** in the variable x is an equation that is equivalent to one that can be written in the form

$$ax + b = 0 \tag{1}$$

where a and b are constants and $a \neq 0$.

EXAMPLE 3 Solving a Linear Equation

Solve $5x - 6 = 3x$.

EXAMPLE Solve $\dfrac{7x+3}{2} - \dfrac{9x-8}{4} = 6$.

$$4\left(\frac{7x+3}{2} - \frac{9x-8}{4}\right) = 4(6)$$

$$4 \cdot \frac{7x+3}{2} - 4 \cdot \frac{9x-8}{4} = 24$$

$$2(7x+3) - (9x-8) = 24$$

$$14x + 6 - 9x + 8 = 24$$

$$5x + 14 = 24$$

$$5x = 10$$

$$x = 2$$

EXAMPLE 9 Solving a Fractional Equation

Solve $\frac{5}{x-4} = \frac{6}{x-3}$.

Solution:

Strategy We first write the equation in a form that is free of fractions. Then we use standard algebraic techniques to solve the resulting equation.

Multiplying both sides by the LCD, $(x-4)(x-3)$, we have

$$(x-4)(x-3)\left(\frac{5}{x-4}\right) = (x-4)(x-3)\left(\frac{6}{x-3}\right)$$

$$5(x-3) = 6(x-4) \qquad \text{linear equation}$$

$$5x - 15 = 6x - 24$$

$$9 = x$$

EXAMPLE 12 Solving a Radical Equation

Solve $\sqrt{x^2+33} - x = 3$.

$$\sqrt{x^2+33} = x + 3$$

$$x^2 + 33 = (x+3)^2 \qquad \text{squaring both sides}$$

$$x^2 + 33 = x^2 + 6x + 9$$

$$24 = 6x$$

$$4 = x$$

Quadratic Equations

Definition

A ***quadratic equation*** in the variable x is an equation that can be written in the form

$$ax^2 + bx + c = 0 \tag{1}$$

where a, b, and c are constants and $a \neq 0$.

EXAMPLE 1 Solving a Quadratic Equation by Factoring

a. Solve $x^2 + x - 12 = 0$.

Solution: The left side factors easily:

$$(x-3)(x+4) = 0$$

Think of this as two quantities, $x - 3$ and $x + 4$, whose product is zero. **Whenever the product of two or more quantities is *zero*, at least one of the quantities *must* be zero.** This means that either

$$x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$

Solving these gives $x = 3$ and $x = -4$, respectively. Thus, the roots of the original equation are 3 and -4, and the solution set is $\{-4, 3\}$.

EXAMPLE 2 Solving a Quadratic Equation by Factoring

Solve $(3x - 4)(x + 1) = -2$.

Solution: We first multiply the factors on the left side:

$$3x^2 - x - 4 = -2$$

Rewriting this equation so that 0 appears on one side, we have

$$3x^2 - x - 2 = 0$$
$$(3x + 2)(x - 1) = 0$$
$$x = -\frac{2}{3},\ 1$$

EXAMPLE 3 Solving a Higher-Degree Equation by Factoring

a. Solve $4x - 4x^3 = 0$.

Solution: This is called a *third-degree equation*. We proceed to solve it as follows:

$$4x - 4x^3 = 0$$
$$4x(1 - x^2) = 0 \qquad \text{factoring}$$
$$4x(1 - x)(1 + x) = 0 \qquad \text{factoring}$$

Setting each factor equal to 0 gives $4 = 0$ (impossible), $x = 0, 1 - x = 0$, or $1 + x = 0$. Thus,

$$x = 0 \text{ or } x = 1 \text{ or } x = -1$$

so that the solution set is $\{-1, 0, 1\}$.

Quadratic Formula

The roots of the quadratic equation $ax^2+bx+c=0$, where a, b, and c are constants and $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

EXAMPLE 6 A Quadratic Equation with Two Real Roots

Solve $4x^2 - 17x + 15 = 0$ by the quadratic formula.

Solution: Here $a = 4$, $b = -17$, and $c = 15$. Thus,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-17) \pm \sqrt{(-17)^2 - 4(4)(15)}}{2(4)}$$

$$= \frac{17 \pm \sqrt{49}}{8} = \frac{17 \pm 7}{8}$$

The roots are $\frac{17+7}{8} = \frac{24}{8} = 3$ and $\frac{17-7}{8} = \frac{10}{8} = \frac{5}{4}$.

EXAMPLE 7 A Quadratic Equation with One Real Root

Solve $2 + 6\sqrt{2}y + 9y^2 = 0$ by the quadratic formula.

Solution: Look at the arrangement of the terms. Here $a = 9$, $b = 6\sqrt{2}$, and $c = 2$. Hence,

$$y = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-6\sqrt{2} \pm \sqrt{0}}{2(9)}$$

Thus,

$$y = \frac{-6\sqrt{2}+0}{18} = -\frac{\sqrt{2}}{3} \quad \text{or} \quad y = \frac{-6\sqrt{2}-0}{18} = -\frac{\sqrt{2}}{3}$$

Therefore, the only root is $-\frac{\sqrt{2}}{3}$.

EXAMPLE 9 **Solving a Quadratic-Form Equation**

Solve $\frac{1}{x^6}+\frac{9}{x^3}+8=0$.

Solution: This equation can be written as

$$\left(\frac{1}{x^3}\right)^2+9\left(\frac{1}{x^3}\right)+8=0$$

$$w^2+9w+8=0$$

$$(w+8)(w+1)=0$$

$$w=-8 \quad \text{or} \quad w=-1$$

$$\frac{1}{x^3}=-8 \quad \text{or} \quad \frac{1}{x^3}=-1$$

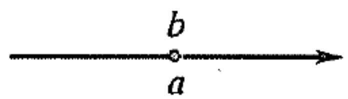
$$x^3=-\frac{1}{8} \text{ or } x^3=-1$$

$$x=-\frac{1}{2} \text{ or } x=-1$$

Linear Inequalities

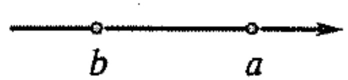
Definition

An ***inequality*** is a statement that one quantity is less than, or greater than, or less than or equal to, or greater than or equal to, another quantity.



$a = b$

$a < b$, a is less than b
$b > a$, b is greater than a



$a > b$, a is greater than b
$b < a$, b is less than a

Rules for Inequalities

1. If $a < b$, then $a + c < b + c$ ***and*** $a - c < b - c$.

For example, $7 < 10$ so $7 + 3 < 10 + 3$.

2. If $a < b$ and $c > 0$, then $ac < bc$ and $\dfrac{a}{c} < \dfrac{b}{c}$.

For example, $3 < 7$ and $2 > 0$ so $3(2) < 7(2)$ and $\frac{3}{2} < \frac{7}{2}$.

3. If $a < b$ and $c < 0$, then $a(c) > b(c)$ and $\dfrac{a}{c} > \dfrac{b}{c}$.

For example, $4 < 7$ and $-2 < 0$ so $4(-2) > 7(-2)$ and $\frac{4}{-2} > \frac{7}{-2}$.

5. If $0 < a < b$ or $a < b < 0$, then $\dfrac{1}{a} > \dfrac{1}{b}$.

For example, $2 < 4$ so $\frac{1}{2} > \frac{1}{4}$ and $-4 < -2$ so $\frac{1}{-4} > \frac{1}{-2}$.

6. If $0 < a < b$ and $n > 0$, then $a^n < b^n$. If $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$.

For example, $4 < 9$ so $4^2 < 9^2$ and $\sqrt{4} < \sqrt{9}$.

EXAMPLE 1 Solving a Linear Inequality

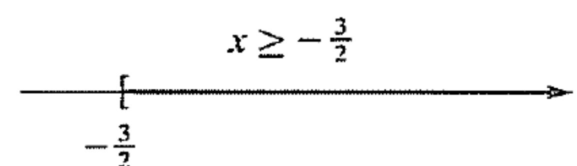
Solve $3 - 2x \leq 6$.

Solution:

$$3 - 2x \leq 6$$

$$-2x \leq 3 \qquad \text{Rule 1}$$

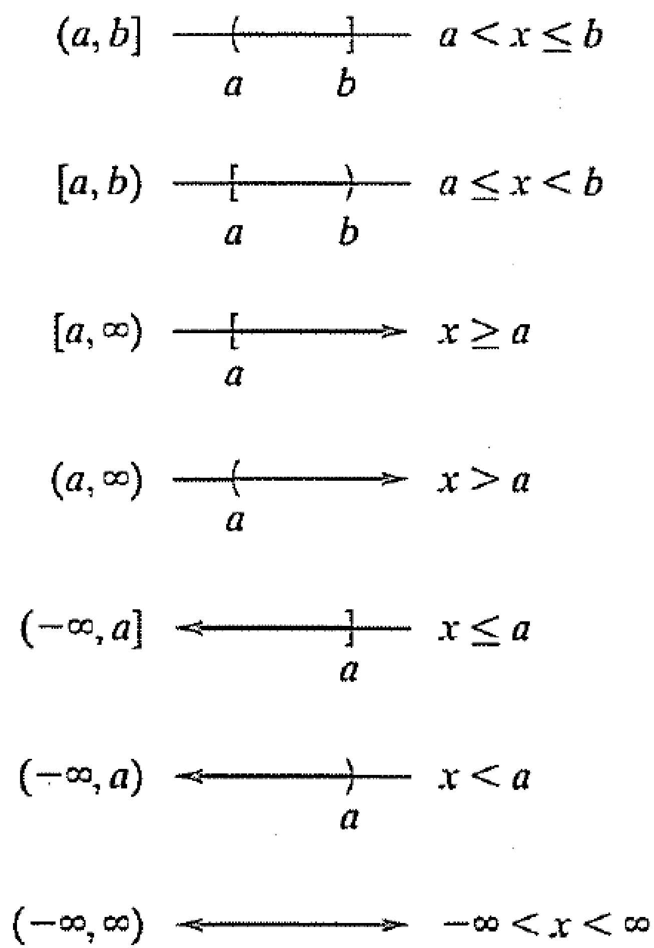
$$x \geq -\frac{3}{2} \qquad \text{Rule 3}$$



The solution is $x \geq -\frac{3}{2}$, or, in interval notation, $[-\frac{3}{2}, \infty)$.

FIGURE 1.14 The interval $[-\frac{3}{2}, \infty)$.

FIGURE 1.12 Closed and open intervals.



Definition

The ***absolute value*** of a real number x, written $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

EXAMPLE 1 Solving Absolute-Value Equations

a. Solve $|x - 3| = 2$.

Solution: This equation states that $x - 3$ is a number 2 units from 0. Thus, either

$$x - 3 = 2 \quad \text{or} \quad x - 3 = -2$$

Solving these equations gives $x = 5$ or $x = 1$.

b. Solve $|7 - 3x| = 5$.

Solution: The equation is true if $7 - 3x = 5$ or if $7 - 3x = -5$. Solving these equations gives $x = \frac{2}{3}$ or $x = 4$.

c. Solve $|x - 4| = -3$.

Solution: The absolute value of a number is never negative, so the solution set is $\emptyset$.

Inequality ($d > 0$)	Solution
$\|x\| < d$	$-d < x < d$
$\|x\| \le d$	$-d \le x \le d$
$\|x\| > d$	$x < -d$ or $x > d$
$\|x\| \ge d$	$x \le -d$ or $x \ge d$

EXAMPLE 2 Solving Absolute-Value Inequalities

a. Solve $|x - 2| < 4$.

Solution: The number $x - 2$ must be less than 4 units from 0. From the preceding discussion, this means that $-4 < x - 2 < 4$. We can set up the procedure for solving this inequality as follows:

$$-4 < x - 2 < 4$$
$$-4 + 2 < x < 4 + 2 \qquad \text{adding 2 to each member}$$
$$-2 < x < 6$$

Thus, the solution is the open interval $(-2, 6)$. This means that all numbers between -2 and 6 satisfy the original inequality. (See Figure 1.20.)

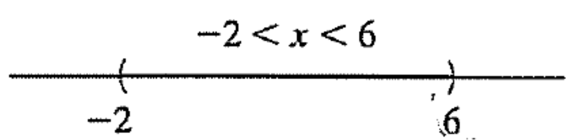


FIGURE 1.20 The solution of $|x - 2| < 4$ is the interval $(-2, 6)$.

EXAMPLE 3 Solving Absolute-Value Inequalities

a. Solve $|x+5| \geq 7$.

Solution: Here $x+5$ must be *at least* 7 units from 0. Thus, either $x+5 \leq -7$ *or* $x+5 \geq 7$. This means that either $x \leq -12$ *or* $x \geq 2$. Thus, the solution consists of two intervals: $(-\infty, -12]$ and $[2, \infty)$. We can abbreviate this collection of numbers by writing

$$(-\infty, -12] \cup [2, \infty)$$

where the connecting symbol $\cup$ is called the *union* symbol. (See Figure 1.21.) More formally, the **union** of sets A and B is the set consisting of all elements that are in either A or B (or in both A and B).

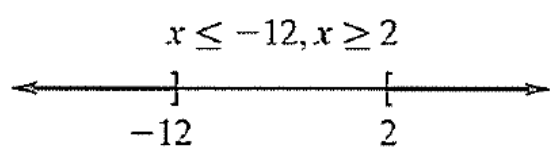


FIGURE 1.21 The union $(-\infty, -12] \cup [2, \infty)$.

Properties of the Absolute Value

Five basic properties of the absolute value are as follows:

1. $|ab| = |a| \cdot |b|$
2. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
3. $|a - b| = |b - a|$
4. $-|a| \leq a \leq |a|$
5. $|a + b| \leq |a| + |b|$

EXAMPLE **Properties of Absolute Value**

a. $|(-7) \cdot 3| = |-7| \cdot |3| = 21$

b. $|4 - 2| = |2 - 4| = 2$

c. $|7 - x| = |x - 7|$

d. $\left|\dfrac{-7}{3}\right| = \dfrac{|-7|}{|3|} = \dfrac{7}{3}; \left|\dfrac{-7}{-3}\right| = \dfrac{|-7|}{|-3|} = \dfrac{7}{3}$

e. $\left|\dfrac{x-3}{-5}\right| = \dfrac{|x-3|}{|-5|} = \dfrac{|x-3|}{5}$

f. $-|2| \leq 2 \leq |2|$

g. $|(-2) + 3| = |1| = 1 \leq 5 = 2 + 3 = |-2| + |3|$